Exam Probability Theory (202001233) in M4-TCS/BIT (14-06-2024, 13.45-15.45 hr.)

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Last Name, First Name: _____

Student ID: _____

- This exam consists of 8 exercises.
- This document is your question as well as your answer paper.
- Write your answer in the designated boxes - try to keep your writing within the box.
- An extra box on the last page is given, in case you need extra space to finish a question. You must write the question number when you use the extra space of last page.
- Additional scratch paper is available for your convenience (not graded!)
- The formula sheet and the probability tables are provided separately. ***You must not write on them and should return them after the exam is over***.
- An ordinary calculator is allowed, not a programmable one (GR).
- Report numerical values to 3 decimal places where appropriate and unless mentioned otherwise.
- Please make your handwriting legible. Write only with blue or black ink. If you are using pencil, make sure that you use dark colour - light colour writing are very difficult to read. If we cannot read your writing, we shall not award points.

Do not use this page for computation or answer your question.
Only write your name and student ID in the designated space.

1	2	3	4	5	6	7	8	Total
2	4	6	1	2	7	11	7	40

Grade $= 1 + 9 \times \dfrac{\textbf{total score}}{40}$.

Part 1.

You do not have to show your works for the exercises of Part 1.

Exercise 1. [**2 points**. *All 5 parts correct: 2pt, 4 parts correct: 1pt, 3 parts correct: 0.5pt, otherwise 0 pt.*]
We have the following information about events A and B:

$$P(A \cap B) = 0.38,\ P(A \cap \overline{B}) = 0.22,\ P(\overline{A} \cap B) = 0.15,\ P(\overline{A} \cap \overline{B}) = 0.25.$$

For each of the following statements, write clearly either **True** or **False** inside the boxes:

a. Events A and B are independent. ☐

b. $P(A \mid B) < P(B \mid A)$. ☐

c. $P(A \mid \overline{B}) > P(B \mid \overline{A})$. ☐

d. $P(\overline{A} \cup \overline{B}) = 0.87$. ☐

e. $P(\overline{A \cup B}) = 0.25$. ☐

Exercise 2. [$4 \times 1 =$ **4 points**]
A shipment of six parrots from Brazil includes two parrots with a potentially fatal disease. As usual, the U.S. Customs Office at the shipment's point of entry randomly samples three parrots and tests them for disease. Let the random variable X be the number of healthy parrots in the sample. Find the probability distribution of X.

a. If S_X represents the range of values X can take, then

A. $S_X = \{0, 1, 2, 3\}$, **B.** $S_X = \{1, 2, 3\}$, **C.** $S_X = \{0, 1, 2\}$, **D.** $S_X = \{0, 1, \ldots, 10\}$.

Answer: ☐

b. $P(X = 2) =$ ☐ [Answer to **3 decimal** places]

c. $E(X) =$ ☐ [Answer to **3 decimal** places]

d. $\mathrm{Var}(X) =$ ☐ [Answer to **3 decimal** places]

Exercise 3. [$4 \times 1 + 2 =$ **6 points**]
The following table displays the joint probability distribution of X and Y:

	$Y = y$		
$X = x$	0	1	2
0	0.13	0.16	0.07
1	0.08	0.14	0.04
2	0.22	0.03	0.13

Answer the following to **3 decimal places**:

a. $P(Y = 1) =$ ☐

b. $P(X = 0 \,|\, Y = 1) =$ ☐

c. $P(X > Y) =$ ☐

d. $P(X > Y \,|\, Y = 1) =$ ☐

e. It is also given that $E(X) = 1.02$, $E(X^2) = 1.78$, $E(Y) = 0.81$, $E(Y^2) = 1.29$, $E(XY) = 0.8$.

Then correlation coefficient, $\rho(X, Y) =$ ☐

Exercise 4. [**1 point**]
Let X and Y be two random variables. Which of the following statements is true?

A. $P(X > 5 \text{ and } Y > 5) = P(X + Y > 10)$ is always true.

B. $P(X > 5 \text{ and } Y > 5) \geq P(X + Y > 10)$ is always true.

C. $P(X > 5 \text{ and } Y > 5) \leq P(X + Y > 10)$ is always true.

D. One cannot decide about A, B and C without further information about probability distributions of X and Y.

Answer: ☐

Exercise 5. [$2 \times 1 =$ **2 points**]
The amount of time you have to wait for a green light at a particular stoplight is uniformly distributed between zero and two minutes.

a. What is the probability that you have to wait more than 30 seconds for the green light? ☐

b. 80% of the time, how many seconds you have to wait *at least* before the light changes to green? ☐

Part 2

You must motivate your answers for the exercises of Part 2.

Exercise 6. [**7 points**]
In a country 40% of the employees work for large companies (> 100 employees): 80% of these employees have a company pension, whereas only 30% of the employees of small companies (≤ 100 employees) have a company pension.

a. What is the probability that an arbitrary employee in this country has a company pension?

Answer this question by first defining relevant events, expressing the given probabilities in these events and using the rules of probability to compute the requested probability. [3pt]

Exercise 6 contd.

b. What is the chance that an employee with a company pension is working for a small company? [2pt]

c. What is the probability that an employee **without** a company pension is working for a large company? [2pt]

Exercise 7. [**11 points**]

Based on research on the length of stay (in seconds) of a customer in a specific shop, the researchers found the following model (given by the density function) of the length of stay X:

$$f_X(x) = \begin{cases} 4x^{-5}, & \text{if } x > 1, \\ 0, & \text{if } x \leq 1. \end{cases}$$

Answer the following (**motivate your answers**):

a. Compute $P(X > 2)$. [2pt]

b. Compute $E(X)$ and $\mathrm{Var}(X)$. [3pt]

Exercise 7 contd.

c. Determine the range (S_Y) of $Y = \dfrac{1}{X^2}$ and the density function $f_Y(y)$ for any $y \in S_Y$. [4pt]

d. Determine $E(Y)$, where $Y = \dfrac{1}{X^2}$. [2pt]

Exercise 8. [**7 points**]

In the elevators of a high building, a sign tells us: "Maximum 1200 kg or 16 persons". So, for instance, if 16 persons all have a weight of at most 75 kg, the elevator is not overloaded ($16 \times 75 = 1200$).

Assume that the weights of the users of the elevator are independent and all normally distributed with mean 72 kg and standard deviation 6 kg. Let $X_1, X_2, \ldots, X_{16}$ are the weights of 16 persons, who all enter the elevator at the same time. Answer the following with motivation:

a. Compute $P(X_1 \leq 75)$, the probability that one person has a weight of at most 75 kg. [2pt]

b. Compute the probability that all 16 persons have a weight of at most 75 kg.
(*Use* 0.736 *for the probability in* **a.**, *ONLY if you did not solve* **a.**) [2pt]

Exercise 8 contd.

c. Compute the probability that the elevator will be overloaded if 16 (arbitrary) persons enter. [3pt]

Extra writing space (can be used for any question as needed). Write the question number which you are answering here.